Mass Conservative Fluid Flow Visualization for CFD Velocity Fields

Zhenquan Li*, Gordon D. Mallinson
Department of Mechanical Engineering, The University of Auckland, New Zealand

Mass conservation is a key issue for accurate streamline and stream surface visualization of flow fields. This paper complements an existing method (Feng et al., 1997) for CFD velocity fields defined at discrete locations in space that uses dual stream functions to generate streamlines and stream surfaces. Conditions for using the method have been examined and its limitations defined. A complete set of dual stream functions for all possible cases of the linear fields on which the method relies are presented. The results in this paper are important for developing new methods for mass conservative streamline visualization from CFD data and using the existing method.

Key Words: Streamline, Stream Surface, Computational Fluid Dynamics (CFD), Mass Conservation, Dual Stream Functions

1. Introduction

Methods for the visualization of three dimensional fluid flows have attracted much attention from different areas such as computer science and engineering. Streamline and stream surface visualization is an important instrument for exploring the properties of a fluid velocity field, especially in three dimensions.

A streamline is everywhere tangential to the velocity field $\mathbf{V}$, i. e., a graph of the solution of:

$$\frac{dx_1}{v_1}=\frac{dx_2}{v_2}=\frac{dx_3}{v_3}$$

if $v_i \neq 0 (i=1, 2, 3)$ or written in vector form as

$$\frac{d\mathbf{X}}{dt}=\mathbf{V} \tag{1}$$

using a Cartesian coordinate system $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ where $\mathbf{V}=\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$. The integration of (1) is usually performed numerically and involves schemes for both the interpolation of the velocity field and time integration (Bunning, 1988; Handscomb, 1984; 1991; Yeung and Pope, 1988). Provided 4th order Runge Kutta schemes (or better) are used for time integration the more significant source of error arises from velocity interpolation (Yeung and Pope, 1988). In particular, failure to conserve mass can produce errors that can not be eliminated by reducing the integration step. These errors can generate artificial effects, such as false spiraling. Thus, the computation is generally a complicated and computationally expensive process (Bunning, 1988; Knight and Malliuson, 1996).

Yih (1957) proved that, for a steady compressible fluid, there exist two stream functions f and g such that the momentum

$$\rho\mathbf{V}=\nabla f \times \nabla g \tag{2}$$

where ρ is the fluid density. The surfaces represented by holding f or g constant are called stream surfaces. Because ∇f and ∇g are perpendicular to the stream surfaces that f and g

* Corresponding Author,
E-mail: zhen.li@auckland.ac.nz
TEL: +64-9-313-7599 ext 4543;
FAX: +64-9-373-7479
Department of Mechanical Engineering, The University of Auckland, New Zealand (Manuscript **Received** August 29, 2001; **Revised** October 10, 2001)

represent, they are also perpendicular to the velocity field $\mathbf{V}$ and hence the cross product $\nabla f \times \nabla g$ is in the direction of the velocity field. Thus the velocity field $\mathbf{V}$ is tangential everywhere to the intersection of the two stream surfaces represented by f and g in (2).

For incompressible flows, we can take $\rho=1$ in (2). The steady compressible fluid flow expressed by (2) obeys the law of mass conservation, as can be seen by taking the divergence of (2):

$$\begin{aligned}\nabla.(\rho \mathbf{V}) &= \nabla(\nabla f \times \nabla g) \\ &= \nabla g.(\nabla \times \nabla f) - \nabla f.(\nabla \times \nabla g) = 0.\end{aligned}$$

A stream surface for a three-dimensional velocity field is a surface across which there is no flow. Stream surfaces can be useful for visualization of such flows because they enable the scientist to isolate part of the flow field for detailed study; the amount of data presented in one visualization is reduced to a manageable quantity. Because no flow crosses the surfaces, restrictions of the visualized data based on stream surfaces can be more physically meaningful than various geometric restrictions.

A CFD velocity field is given at discrete locations in space. It is understood that the discrete velocity field is an approximation to a continuous mass conservative velocity field in the same domain. Although the most appropriate interpolation scheme for visualization is dependent on the connectivity and approximations used by the CFD solver, in the majority of cases such information is not available to the visualization system. Often quite simple interpolation strategies are used, one of the most common being a locally linear interpolation of the velocity. When a mesh consists of tetrahedra the interpolation for each can be defined unambiguously using the values of velocity at the four vertices, [e.g. (Nielson and Jung, 1999)]. For hexahedra, an interpolation scheme can be defined directly or by subdividing each hexahedron into tetrahedra. Generally the definition of the interpolation is ambiguous. For the purpose of this discussion, however, it is sufficient to consider tetrahedral meshes.

Linear interpolation of the velocity over each tetrahedron in a mesh is a mathematical approximation and the resulting field does not necessarily satisfy mass conservation. Mass conservation is a key issue to construct accurate streamlines (Mallinson, 1988). Feng *et al.* (1997) described a technique whereby streamlines and stream surface are generated by mass conservative interpolation schemes based on a similar expression of (2) for the CFD velocity field for incompressible flow. Even though their technique is very useful for the computer visualization of three-dimensional velocity fields, there is an important issue regarding their process for generating a mass conservative velocity field from the linear interpolation of the discrete data that needs more careful handling then in (Feng et al., 1997). There are conditions that must be met for their method to generate correct results in all situations. This paper derives the conditions of validity for the use of their method for steady flows.

2. Construction of Mass Conservative Velocity Fields

If a linear velocity field $\mathbf{V}_l = \begin{pmatrix} V_{l1} \\ V_{l2} \\ V_{l3} \end{pmatrix}$ is the linear interpolation of the values at the four vertexes of tetrahedra in the domain of the velocity field, then it can be given in every tetrahedron as

$$\mathbf{V}_l = \mathbf{A}\mathbf{Y} + \mathbf{B}',$$

where $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and $\mathbf{B}' = \begin{pmatrix} b_1{}', \\ b_2, \\ b_3, \end{pmatrix}$ are constant matrix and vector respectively and $\mathbf{Y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ is the coordinate vector. For steady flow, if the values of the fluid density ρ at the vertices of the grid are given, and assuming $\rho \mathbf{V}_l$ is linear, we can replace $\mathbf{V}_l$ in the above equation by $\rho \mathbf{V}_l$ and the following discussions are the same. For simplicity, however, we will ignore the fluid density ρ, i.e., by assuming $\mathbf{V}(\mathbf{Y})$ is incompressible in the following.

In (Feng et al., 1997), $\mathbf{V}(\mathbf{Y})$ was assumed to be a mass conservative velocity related to $\mathbf{V}_l(\mathbf{Y})$

$$\mathbf{V}(\mathbf{Y})=F(\mathbf{Y})\mathbf{V}_l(\mathbf{Y}), \quad (3)$$

by where F is a scalar function. Because f is a stream function corresponding to $\mathbf{V}(\mathbf{Y})=\begin{pmatrix}V_1\\V_2\\V_3\end{pmatrix}$, it satisfies

$$\frac{\partial f}{\partial y_1}V_1+\frac{\partial f}{\partial y_2}V_2+\frac{\partial f}{\partial y_3}V_3=\nabla f\cdot\mathbf{V}=0. \quad (4)$$

A similar equation exists for g. Substituting (3) into (4) leads to

$$F\left[\frac{\partial f}{\partial y_1}V_{l1}+\frac{\partial f}{\partial y_2}V_{l2}+\frac{\partial f}{\partial y_3}V_{l3}\right]=0. \quad (5)$$

The authors of (Feng et al. 1997) claimed that (5) had the same solution as

$$\frac{\partial f}{\partial y_1}V_{l1}+\frac{\partial f}{\partial y_2}V_{l2}+\frac{\partial f}{\partial y_3}V_{l3}=0 \quad (6)$$

and that it was necessary to consider only the streamlines and stream surfaces for $\mathbf{V}_l$.

The issue that this paper considers is the existence of function F in (3). The conditions for F such that (5) and (6) have the same solutions are that it is not equal to zero and/or infinity in the considered domain. Because there is no apriori information about F, we have to calculate F and then determine the conditions that (5) and (6) are equivalent before the results in (Feng et al. 1997) can be used for the generation of streamlines and stream surfaces.

Because we draw the streamlines cell by cell, it is sufficient to discuss this issue in a single cell. For incompressible flows, mass conservation means that the velocity $\mathbf{V}(\mathbf{Y})$ satisfies

$$\nabla\cdot\mathbf{V}=0.$$

The velocity $\mathbf{V}(\mathbf{Y})$ in (Feng et al. 1997) satisfies

$$\nabla\cdot\mathbf{V}=\nabla\cdot(F\mathbf{V}_l)=0,$$

i.e.,

$$\nabla\cdot\mathbf{V}=\frac{\partial(FV_{l1})}{\partial y_1}+\frac{\partial(FV_{l2})}{\partial y_2}+\frac{\partial(FV_{l3})}{\partial y_3}$$
$$=\frac{\partial F}{\partial y_1}V_{l1}+\frac{\partial F}{\partial y_2}V_{l2}+\frac{\partial F}{\partial y_3}V_{l3}+\mathit{trace}(\mathbf{A})\cdot F=0$$

or more simply

$$\frac{DF}{Dt}=-\mathit{trace}(\mathbf{A})F, \quad (7)$$

where the left hand side is the material derivative and the first term in the right hand side is the trace of matrix $\mathbf{A}$.

From standard algebraic results, every $n\times n$ matrix is similar to the Jordan form

$$\mathfrak{J}'=\mathit{diag}[\mathbf{J}_1, \mathbf{J}_2, \cdots, \mathbf{J}_q],$$

where $\mathbf{J}_i$ is a $k_i\times k_i$ upper triangular Jordan block matrix, and $k_1+k_2+\cdots+k_q=n$, i. e., there is a non-singular $n\times n$ matrix $\mathbf{P}$ such that

$$\mathfrak{J}'=\mathbf{P}^{-1}\mathbf{A}\mathbf{P}.$$

Matrixes $\mathbf{J}_i$ and $\mathbf{P}$ are complex. However, the visualization is in the real domain. Because two matrixes are similar to each other if they are similar to a Jordan form, then we can find all the cases of real matrixes that are similar to all the possible Jordan forms of $\mathbf{A}$. Because the linear transformations of the coordinates are non-singular, the visualization of the streamlines of $\mathbf{V}_l=\mathbf{A}\mathbf{Y}+\mathbf{B}'$ is the same as that of the system

$$\dot{\mathbf{Z}}=\mathfrak{J}\mathbf{Z}+\mathbf{B} \quad (8)$$

where $\mathfrak{J}=\mathbf{P}'^{-1}\mathbf{A}\mathbf{P}'$. Both $\mathfrak{J}$ and $\mathbf{P}'$ are real matrices (called $\mathfrak{J}$ and $\mathbf{A}$ real similar), $\mathbf{Z}=\mathbf{P}'\mathbf{Y}$ and $\mathbf{B}=\mathbf{P}'^{-1}\mathbf{B}'$. For symbol agreement with (Feng et al, 1997), we have to make a translation of (8) such that the final system considered is

$$\dot{\mathbf{X}}=\mathfrak{J}\mathbf{X}. \quad (9)$$

Because the trace of a matrix is invariant under similar transformations and translations, Eq. (7) is unchanged in the coordinate system

$$\mathbf{X}=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix},$$

i. e., the calculation procedure for the scalar function F in (3) is not affected by the transformations to the new coordinate system. The following discussions are for the new coordinate system. The scalar function F in the original coordinate system can be derived by the inverse transformations and the discussions for $F\neq 0$ are similar.

Table 1 Jacobean and dual stream functions for all possible cases of a conservative linear field

Case	Jacobean	Dual stream functions
1	$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -a-b \end{pmatrix}$	$f=\dfrac{x_2}{x_1^{\frac{b}{a}}},\ g=\dfrac{x_3}{x_1^{\frac{-a-b}{a}}}$
2	$\begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & -2a \end{pmatrix}$	$f=b\ln x_3-2a\operatorname{arctg}\dfrac{x_2}{x_1},\ g=-\dfrac{1}{2}\ln x_3-\dfrac{1}{2}\ln(x_1^2+x_2^2)$
3	$\begin{pmatrix} a & 0 & 0 \\ 1 & a & 0 \\ 0 & 0 & -2a \end{pmatrix}$	$f=\dfrac{x_2}{x_1}+\dfrac{\ln x_3}{2a},\ g=a\ln x_3+2a\ln x_1$
4	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$f=x_1,\ g=x_3x_1-\dfrac{x_2^2}{2}$
5	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -a \end{pmatrix}$	$f=x_1-\dfrac{1}{a}\ln x_2-\dfrac{1}{a}\ln x_3,\ g=x_2x_3$
6	$\begin{pmatrix} 0 & b & 0 \\ -b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$f=\dfrac{b}{2}\ln(x_1^2+x_2^2),\ g=x_3$
7	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$f=x_1,\ g=x_3$

3. Conditions for the Existence of the Function F

The following expressions of function F are calculated by solving Eq. (7) and then deleting parameter t in expressions of the solutions through the parametric stream surface as given by (12) in (Feng et al, 1997). These calculations involve some complicated derivations and we only show two examples of the calculations and then list only the resulting expressions. There are two main groups of situations, *trace*$(\mathbf{A})=0$ and *trace*$(\mathbf{A})\neq 0$. The first corresponds to a linear interpolation that is already mass conservative, the second to an interpolation that is not.

3.1 Mass conservative linear field

In this situation, *trace*$(\mathbf{A})=0$: $F=$constant, i.e., (5) and (6) are equivalent each other. That means the results in (Feng et al, 1997) are correct for this case. The dual stream functions listed in Table 1 can be used directly for the generation of streamlines and surfaces without investigating function F. The following calculations for the case that all the eigenvalues of $\mathbf{A}$ are zero and it has multiplicity 3 (case 4 in Table 1) are an example of how to derive the dual stream functions.

If all the eigenvalues of $\mathbf{A}$ are zero and it has multiplicity 3, i. e.,

$$\mathfrak{J}=\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The solutions of (9) for the parametric stream surfaces in the local coordinate system of the cell are

$$\begin{aligned} x_1&=c_1 \\ x_2&=c_1t+c_2 \\ x_3&=\frac{1}{2}c_1t^2+c_2t+c_3 \end{aligned} \tag{10}$$

where $c_i(i=1, 2, 3)$ are constant. Solving for t from the expression of x_2 and then substituting it into the expression of x_1 and making some arrangements, we have

$$x_3-\frac{1}{2}\frac{x_2^2}{c_1}=c_3-\frac{c_2^2}{2c_1} \text{ if } c_1\neq 0 \text{ or } x_1\neq 0$$

or

$$x_1x_3-\frac{x_2^2}{2}=c_1c_3-\frac{c_2^2}{2} \text{ since } x_1=c_1.$$

From the definition of stream functions in Sec. 1 and using the fact that right hand sides of the above equation and $x_1=c_1$ are constant, we can choose the dual stream functions as

$$f=x_1,\ g=x_3x_1-\frac{x_2^2}{2}.$$

If $c_1=0$, the streamline (10) is a straight line parallel to the x_3 axis on the x_2x_3 plane. The intersection of dual stream surfaces $f=0$ and $g=-c^2$(c constant) gives a straight line that is parallel to the x_3 axis on the x_2x_3 plane as well. Therefore intersections of the dual stream surfaces give all the streamlines (10).

3.2 Non conservative linear field

In this situation, $trace(\mathbf{A})\neq 0$. We show an example of how to calculate function F and dual stream functions. These functions can be calculated similarly for all other cases. Table 2 gives the Jacobean, expression of F, dual stream functions and existence conditions for F for all cases.

Consider case 1 when $\mathfrak{J}$ has three real nonzero eigenvalues a, b, $c(a+b+c\neq 0)$, i. e.

$$\mathfrak{J}=\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}.$$

The solutions of (9) for the parametric stream surfaces under the local coordinate system of the cell is

$$\begin{aligned} x_1&=c_1e^{at},\\ x_2&=c_2e^{bt},\\ x_3&=c_3e^{ct}, \end{aligned} \tag{11}$$

and the solution of (7) is

$$F=ce^{-(a+b+c)t}.$$

The dual stream functions can be calculated by solving one of the three equations in (11) for t and then substituting it into the other two. Here we solve the first and then substitute it into second and third expressions and have

$$\frac{x_2}{x_1^{\frac{b}{a}}}=C_1,\ \frac{x_3}{x_1^{\frac{c}{a}}}=C_2\cdot(C_1,\ C_2 \text{ constant})$$

From the definition of dual stream functions we can choose

$$f=\frac{x_2}{x_1^{\frac{b}{a}}},\ g=\frac{x_3}{x_1^{\frac{c}{a}}}.$$

Function $F=ce^{-(a+b+c)}=C(c_1e^{at})^{-1}(c_2e^{bt})^{-1}(c_3e^{ct})^{-1}=C/(x_1x_2x_3)$. ($C$ constant)

All the functions in Table 2 can be calculated in a similar manner. We omit the calculations and list only the results.

4. Discussion

The concepts discussed in this paper can be demonstrated by the swirling flows illustrated in Fig. 1. Both have a Jacobean of the form corresponding that in Case 2 in Table 2 and are classified according to the kind of singularity that exists at (0, 0, 0). For the unstable focus flow $a=1.5$, $b=20$, $c=2$ and for the saddle-spiral $a=1.5$, $b=20, c=-2$. Both have $trace(\mathbf{A})\neq 0$. In each representation, an iso-surface of each dual stream function has been drawn together with a streamline that passes close (for clarity) to their intersection. A streamline on the other side of the $x_3=0$ plane has also been drawn. Both streamlines have been constructed by integrating the non-conservative linear velocity field $\mathbf{V}_l(\mathbf{Y})$. Lines constructed by integrating $\mathbf{V}(\mathbf{Y})$ are identical as implied by the fact that the streamlines in the figure coincide with the intersection of the stream surfaces. The gauge transformation generated by the function F effectively adjusts the speed of the fluid to ensure mass conservation while retaining the same flow structure.

As indicated by the entry for case 2 in Table 2, F will not exist on the $x_3=0$ plane or along the axis of rotation $x_1=x_2=0$. Note each representation in Fig. 1 is for a continuous linear field. Each cell in a CFD field will contain its own linear field. As a stream surface is propagated from cell to cell by a process such as the one

Table 2 Jacobean, function F, dual stream functions and conditions for the existence of F for all possible cases of a non conservative linear field

Case	Jacobean	F	Dual stream functions	Conditions
1	$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$	$F=C/(x_1x_2x_3)$	$f=\frac{x_2}{x_1^{\frac{b}{a}}}\ g=\frac{x_3}{x_1^{\frac{c}{a}}}$	Any one of x_1, x_2 and x_3 does not go through zero
2	$\begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & c \end{pmatrix}$	$F=\frac{C}{2(x_1^2+x_2^2)x_3}$	$f=b\ln x_3+c\,\mathrm{arctg}\frac{x_2}{x_1}$ $g=\frac{a}{c}\ln x_3-\frac{1}{2}\ln(x_1^2+x_2^2)$	x_3 or both x_1 and x_2 do not go through zero
3	$\begin{pmatrix} a & 0 & 0 \\ 1 & a & 0 \\ 0 & 0 & c \end{pmatrix}$	$F=\frac{C}{x_1^2x_3}$	$f=\frac{x_2}{x_1}-\frac{\ln x_3}{c}$ $g=a\ln x_3-c\ln x_1$	x_1 or x_3 does not go through zero
4	$\begin{pmatrix} a & 0 & 0 \\ 1 & a & 0 \\ 0 & 1 & a \end{pmatrix}$	$F=\frac{C}{x_1^3}$	$f=\frac{1}{2}\left(\frac{x_2}{x_1}\right)^2-\frac{x_3}{x_1}$ $g=a\left(\frac{x_2}{x_1}-\frac{1}{a}\ln x_1\right)$	x_1 does not go through zero
5	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}$	$F=\frac{C}{x_2x_3}$	$f=x_1-\frac{1}{a}\ln x_2+\frac{1}{b}\ln x_3$ $g=b\ln x_2-a\ln x_3$	Both x_1 and x_2 do not go through zero
6	$\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$F=\frac{C}{x_1}$	$f=ax_2$ $g=x_3$	x_1 does not go through zero
7	$\begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$F=\frac{C}{x_1^2+x_2^2}$	$f=\frac{b}{2}\ln(x_1^2+x_2^2)-a\arctan\left(\frac{x_1}{x_2}\right)$ $g=x_3$	Both x_1 and x_2 do not go through zero
8	$\begin{pmatrix} a & 0 & 0 \\ 1 & a & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$F=\frac{C}{x_1^2}$	$f=\frac{ax_2}{x_1}-\ln x_1$ $g=x_3$	x_1 does not go through zero

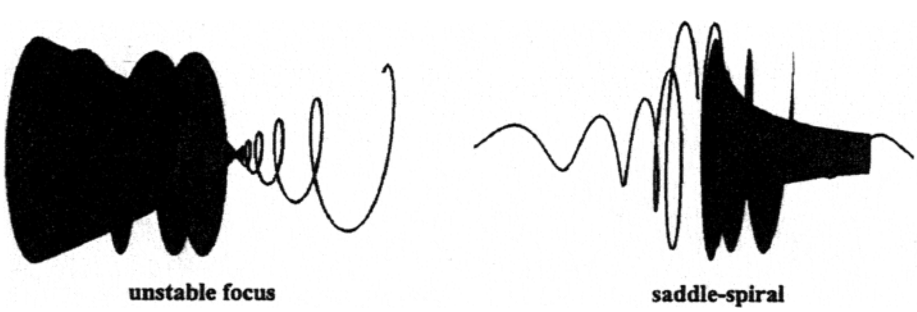

Fig. 1 Dual stream functions and streamlines near focus and saddle singularities. In both cases the underlying linear field is non-conservative

described in (Feng et al, 1997) it may enter a cell unfavourably with respect to the conditions required to ensure that F exists. Hence it is important that the conditions derived in this paper are understood and tested for during stream surface construction.

5. Conclusion

The technique given in (Feng et al, 1997) is a useful visualization scheme for CFD data. Using the results of this paper, the method given in

(Feng et al, 1997) should be modified to ensure that the function F that permits mass conservative streamline construction to be achieved actually exists in all cases. However, if the scalar function F equals to zero or infinity at a point in original space, the method given in (Feng et al, 1997) can not be used for generating mass conservative streamline and new methods may be necessary.

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